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Renormalization of the Potts model

Daniel J Amit†

Service de Physique Théorique, Centre d'Etudes Nucléaires de Saclay, BP n° 2,91190 Gif-sur-Yvette, France

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Abstract. We study the Potts model with a general number of states. First, we discuss the situation in which the Landau theory leads to a first order transition, but which does show fixed points of the renormalization group. Here, there are many questions which need further clarification.

Then we describe, rather pedagogically, the logic behind the application of dimensional regularization to critical phenomena. We argue that this is a particularly natural approach. This technique is then applied to the Potts model, for which the critical exponents are computed to $O[(6-d)^2]$, when there is a fixed point. The one state results, which correspond to the percolation problem, are compared with other calculations and with numerical simulation.

1. Introduction

In the present paper we treat the Ashkin–Teller–Potts model (Ashkin and Teller 1943, Potts 1952) using t'Hooft and Veltman's (1972) technique of dimensional regularization combined with minimal renormalization. We have two objectives. The first is to present a restatement of the technique in order to show that it is a natural way of applying field theory to critical phenomena. The main conceptual advantage is that one avoids discussing ultraviolet divergences in a theory which has a cut-off and in which we are seeking infrared behaviour. At the same time all the machinery erected for the treatment of the ultraviolet is preserved and is applicable.

The second objective is to study the s state Potts model as a possible candidate for a continuous phase transition engendered by a trilinear coupling in the Lagrangian. This model has recently been attracting an increasing amount of attention, due to the fact that it can describe a very rich class of physical situations. In addition, its special symmetry gives rise to some interesting coupling constant flows, even in the absence of the trilinear term (Zia and Wallace 1975).

The three state version of the model can describe the transition of a liquid crystal from its nematic to its isotropic phase (De Gennes 1969, Alexander 1974), or the transition of a cubic crystal into a tetragonal phase (Weger and Goldberg 1973). The two state version is of course the Ising model. The Potts model with a single state can describe the critical behaviour of bond percolation, while the limit of zero states is related to the electrical resistor network (Fortuin and Kasteleyn 1972, Baxter 1973, Harris *et al* 1975).

† On leave from the Racah Institute of Physics, Hebrew University, Jerusalem.

As is indicated by the original application of Ashkin and Teller, the same model with s states is a candidate for describing many systems in which one has s states locally, but the energy of any given configuration depends only on whether pairs of sites are in the same state or in different states.

As was shown by Golner (1973) and Zia and Wallace (1975), such a system can be described by a continuous field theory with

$$n = s - 1 \tag{1.1}$$

real fields. If the n fields are taken to be real, the statistical weight (or the Lagrangian) is symmetric under the discrete group which maps the n dimensional tetrahedron on itself. Such a Lagrangian includes, in general, trilinear terms. Consequently, in the Landau approximation of the theory one finds a first order transition (De Gennes 1971). If this was also the behaviour of the full theory, then renormalization group analysis would be of little help, and one would not be able to appeal to universality.

Thus, the first question which attracts attention is the order of the transition. Baxter (1973) has proved rigorously that in two dimensions the discrete Potts model has a continuous phase transition for $s \leqslant 4$, and a first order transition for $s > 4$. Straley (1974) has argued on the basis of series expansions that the three state model has a continuous transition in three dimensions also. But these results are in contradiction with Golner (1973), Amit and Scherbakov (1974), and Enting and Domb (1975) who find that in three dimensions the transition is a first order one.

On the experimental side, for first order transitions, both the nematic to isotropic transition in liquid crystals and the martensitic transition in β tungsten (Weger and Goldberg 1973) have much smaller discontinuities than those expected from a mean field calculation. The sizes of the susceptibilities, in the generalized sense, near those two transitions are far from negligible.

Work on the percolation problem has also indicated a continuous transition (Kirkpatrick 1975). So once the impact of Fortuin and Kasteleyn's argument, showing that percolation can be described by a limiting case of the Potts model which has a Hamiltonian, was realized by Harris *et al* (1975), they applied the renormalization group, among other techniques, to the study of percolation. The fact that the continuous field theory includes a trilinear term implies directly that the critical number of dimensions separating Gaussian from non-Gaussian behaviour is $d_c = 6$† (see also § 3 below). The natural expansion parameter is then

$$\epsilon = 6 - d. \tag{1.2}$$

Harris *et al* (1975) studied the recursion relations of Wilson and Fisher (1972) to first order in ϵ. A fixed point was found, at this order, for the percolation problem. The critical exponents were calculated to this order giving rather unimpressive agreement with the simulation studies of Kirkpatrick (1975).

If the renormalization group equations are studied for a general number of states, one finds that the value of the trilinear coupling constant, at the fixed point, tends to infinity as $s \to 10/3$ from below‡. For $s > 10/3$ a fixed point implies an imaginary

† Toulouse (1974) has suggested that the critical dimensionality for the percolation problem is 6, before the equivalence to the continuous field theory was appreciated. The logic of Toulouse's conjecture was recently put in question, by A Coniglio (private communication), by pointing out that it is not clear that on a Bethe lattice $\eta = 0$ and $\nu = 1/2$.

‡ This behaviour was first communicated to me by Dr E Brézin, and served as a stimulus for the calculation to $O(\epsilon^2)$.

coupling constant. This clearly limits the validity of the calculation as one approaches the critical value of s. Above $s = 10/3$ the result is interpreted as the absence of a fixed point and thus no continuous transition.

As the Landau theory predicts a first order transition, the discovery of a fixed point created unease. It is commonly thought that the renormalization group can, at most, decorate a continuous transition which already exists in the tree approximation. Harris *et al* (1975) have tried to deal with this issue by imposing a constraint on the order parameter in the Landau theory. It was decided that in the ordered phase the order parameter must lie in the positive direction along the hyper-tetrahedral lines—from the centre to the vertices—if the trilinear coupling constant g_{30} is negative. Normally, such a constraint will be rather harmless since these directions are just those along which the minimum in the free energy—at which the symmetry is discontinuously broken—will lie. But in the Potts model the trilinear term can change its sign because of its tensorial structure. This happens at $s = 2$. Then the constraint imposed for $g_{30} < 0$ eliminates the first order transition and one has to lower the temperature until a second order transition appears, in the positive direction, at the mean field transition temperature of the $g_{30} = 0$ model. This apparently solves two difficulties at once: that of passing from a first order transition to a second order one, and that of the peculiar behaviour near $s = 10/3$.

We find this argument rather unconvincing. It is one thing to impose a constraint on the way one interprets the results of the Landau theory, but it is quite another thing to say that the field theory based on a Lagrangian which knows nothing of the constraint, will describe the small modifications around the selected minimum. In other words, one can consider a situation in which there is an acceptable minimum in the free energy away from the origin. One can then proceed to carry renormalization group transformations in the *symmetric state* and a fixed point may be found: this would be the case for the Potts model when $2 < s < 10/3$. Here there are good reasons to doubt the results, since they would indicate that the renormalization group is insensitive to minima which are a finite distance away. The only difference in the case $s < 2$ is that an argument exists for rejecting the first minimum which appears. But this interpretation of the results of the theory does not improve its reliability in producing a scale invariant symmetric theory in the presence of a deeper, non-symmetric minimum. We do not suggest an answer to this question, which we feel should receive more attention.

The extension of the calculation to second order in ϵ was intended to throw some light on the approach to the critical value of s, and to try and reach a better quantitative agreement with the simulation studies. As this work was being completed, we received the work of Priest and Lubensky (1976) who have studied the Wilson–Fisher recursion relations to order ϵ^2. Nevertheless we feel that the present calculation is of interest. The rather complicated calculation is carried out here by a different method, which seems to us simpler, and includes many internal checks at intermediate stages (see for example § 4 and appendix 2). The method is one of renormalization by minimally subtracting singularities of a dimensionally regularized theory (t'Hooft and Veltman 1972, t'Hooft 1973, Gross 1975, Lawrie 1975). We use this opportunity to argue that this is a very natural approach to the application of field theory to critical phenomena.

The results we find are in qualitative agreement with Priest and Lubensky as far as the approach to $s = 10/3$ is concerned but the exponent γ is significantly different from their result. This difference is big enough to make our results agree with those of Kirkpatrick, to within his errors, down to $d = 4$. The situation with regards to β is much worse. The ϵ^2 term is very small. The discrepancy between the first order result and the

simulation is not improved. We discuss the differences from Priest and Lubensky and the remaining discrepancies in β in § 9. See also the epilogue in § 10.

2. Formulation of the problem

We consider the Potts model with $n+1$ states in the representation proposed by Zia and Wallace (1975). The Landau–Ginsburg free energy density is written as

$$\mathscr{L}=\frac{1}{2}(\nabla\boldsymbol{\phi})^2+\frac{1}{2}m_0^2\boldsymbol{\phi}^2+\frac{1}{3!}g_{30}Q_{ijk}\phi_i\phi_j\phi_k+\frac{1}{4!}g_{40}T_{ijkl}\phi_i\phi_j\phi_k\phi_l \tag{2.1}$$

where $\boldsymbol{\phi}$ is a real field with n components, whose Fourier decomposition includes wavenumbers bounded by a cut-off Λ. The tensor

$$Q_{ijk}=\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha \tag{2.2}$$

is the trilinear hyper-tetrahedral invariant constructed from the $n+1$ vectors $\boldsymbol{e}^\alpha$, which have the property

$$e_i^\alpha e_i^\beta=(n+1)\delta^{\alpha\beta}-1. \tag{2.3}$$

Since we are interested in the limit $n\to 0$ we have changed the normalization of the vectors by a factor of $\sqrt{n}$ relative to that of Zia and Wallace. It follows from (2.3) that

$$\sum_{\alpha=1}^{n+1} e_i=0 \tag{2.4}$$

and

$$e_i^\alpha e_j^\alpha=(n+1)\delta_{ij}. \tag{2.5}$$

Summation over repeated indices is implied. One can write an explicit representation for the tensor T (Zia and Wallace 1975), but we will not need it here.

Averages, correlation functions, vertex functions etc, are defined using

$$\exp-\int \mathrm{d}x\,\mathscr{L}[\boldsymbol{\phi}(x)]$$

as the relative statistical weight of the distribution $\boldsymbol{\phi}(x)$, in the conventional way.

The question is whether the field theory, as specified by (2.1), can be scale invariant; namely, whether the theory can have a non-trivial fixed point. Since the existence of a fixed point ensures the homogeneous behaviour of the various thermodynamic quantities, scaling holds, and other exponents can be obtained. For a review of this ideology, see Brézin *et al* (1975).

3. General considerations

3.1. Infrared singularities and dimensional regularization

As usual, critical behaviour at a phase transition is generated by infrared singularities in Feynman integrals, which are cut off at high momentum. In the absence of infrared

singularities the theory would be Gaussian, at least to leading order in the low momentum region. Clearly, the region around the surface of the integration domain contributes nothing to the singular dependence on the low external momenta. Thus, in any integral which is ultraviolet divergent we can perform an integration by parts, using 'partial p' (t'Hooft and Veltman 1972) obtaining a surface term, and an integral which is less ultraviolet divergent by at least one power (see appendix 1). This procedure can be repeated until the integral is convergent at high momentum. The limits of integration can then be extended to infinity. The difference between the convergent cut-off integral and the integral with infinite limits as well as the surface terms—originating from the integrations by parts—are all analytic functions of the external momenta. The same is true if one considers the graphs at zero external momenta but with a finite small mass (inverse correlation length).

The analytic terms, which depend on the cut-off can be disregarded as far as the asymptotic critical behaviour is concerned, and the remaining integral can be performed. But now one realizes, following t'Hooft and Veltman (1972), that the result is in general identical to that obtained by computing the original integral in its dimensionally regularized form. (See example in appendix 1.)

This procedure is quite general and its usefulness is even more striking when one considers the graphs of the two point function, which are ultraviolet divergent even for $d<6$ (see for example t'Hooft and Veltman 1972, § 4). In that case the integration by parts—'partial p'—makes a p^2 dependence explicit and leaves in the integrals only the singular dependence on p. This also takes place in the dimensional regularization.

The procedure for calculating any graph is to consider the number of dimensions as a complex variable, to assemble groups of denominators by Feynman parameters, and then, paying no attention to questions of ultraviolet convergence, use formulae (A.3) and (A.7) to obtain the integral as a function of the external momenta and the dimensionality. Experience helps in finding the most efficient order in which to group the denominators.

3.2. The relevant operator

Within the framework of perturbation theory the most relevant operator in $\mathscr{L}$ is identified as the one producing the strongest infrared divergencies. Equivalent identification can, of course, be made within the recursion relation scheme (Wilson and Kogut 1974). Here we note that a general graph of a vertex function with E external legs, n_3 trilinear interactions and n_4 quadrilinear interactions has L loops, or free integrations, given by

$$L=\tfrac{1}{2}n_3+n_4-\tfrac{1}{2}E+1. \tag{3.1}$$

Once the dimensional regularization has been introduced, the momentum dependence of a graph can be simply deduced by dimensional analysis, since the integrals become homogeneous functions of the external momenta (Symanzik 1973). If the scale of the external momenta is P, the regularized graph G will behave as:

$$G\sim P^{-2E+6+2n_4+L(d-6)} \tag{3.2}$$

where n_3 was eliminated using equation (3.1). Thus any increase in the number of four-interactions, at a fixed number of loops and external legs, weakens the singularity at low P by two complete powers. The conclusion is that *if a zero mass theory exists* only the trilinear interaction is relevant.

3.3. The critical number of dimensions

The critical number of dimensions is the one at which the fluctuations (infrared divergences) become unimportant. It can be determined from the condition that the dependence of the various vertex functions on the momentum scale be equal to their canonical engineering dimension, since the latter is the scale dependence of the Gaussian (free) theory.

The canonical dimension of an E point vertex function, in units of momentum, is $E+d-\frac{1}{2}Ed$. In a theory with trilinear interactions only, the dependence of this function on the momentum scale is given by (3.2) to be $6-2E+L(d-6)$.

The difference between the two is

$$(\tfrac{1}{2}E+L-1)(d-6)=\tfrac{1}{2}n_3(d-6). \tag{3.3}$$

Thus for $d=6$, the two become equal, and

$$d_c=6, \tag{3.4}$$

as was originally conjectured by Toulouse (1974).

This is of course also the number of dimensions at which the theory is renormalizable, since the ultraviolet divergences become independent of the order in perturbation theory.

3.4. Poles, renormalization, and relevant vertex functions

As one approaches the critical number of dimensions, $d=6$, graphs develop poles, such as the one appearing in $\Gamma[3-(d/2)]$, equation (A.4) in appendix 1. Usually, these poles are ascribed to logarithmic ultraviolet divergences. However in the present interpretation we never have to refer to ultraviolet divergences, since we are considering a cut-off theory.

Instead, we notice that since the dimensionally regularized graphs are identical to the expressions obtained by performing partial integrations, such poles will occur whenever the numbers of powers of momentum in the numerator and in the denominator of the integrand are equal; we will call these logarithmic integrands. This condition is equivalent to that of logarithmic ultraviolet divergence, of course. Only the interpretation, which stays within the cut-off theory, is different.

The origin of these poles can be traced in a simple example. Consider the integral

$$I_\alpha(a)=\int_0^1 \mathrm{d}x/(x+a)^\alpha \tag{3.5}$$

with $a>0$. This integral is clearly finite for any α. If $I_\alpha(a)$ is now calculated by 'partial p', that is, by inserting $\mathrm{d}x/\mathrm{d}x=1$ in (3.5) we find:

$$I_\alpha(a)=\frac{1}{1-\alpha}\frac{x}{(x-a)^\alpha}\bigg|_0^1-\frac{\alpha}{1-\alpha}\int_0^1\frac{a}{(x+a)^{\alpha+1}}\,\mathrm{d}x. \tag{3.6}$$

If the surface term, the first term in equation (3.6), were to be discarded, a pole would appear as $\alpha\to 1$. This is a manifestation of the fact that when one has the same power in the numerator and in the denominator, the integral cannot be calculated by partial integration, except as a limit when $\alpha\to 1$. The integral stays finite, and it is the role of the surface term to cancel the singularity in the second term.

When surface terms are omitted, as was suggested in § 3.1 above, or in appendix 1, the singularities which appear as poles in $\alpha = 6-d$ have to be cancelled by some other means. But, as the simple example shown above indicates, nothing magical occurs; no ultraviolet divergences appear where there were none.

On subtracting the singularity one does not want to reconstitute the entire surface term, which was dropped. Instead one can, for example, consider the second term in (3.6) as the regularized value of $I(a)$ and subtract it from the value of the same integral at some fixed value a', of a. Alternatively, one can simply set $\Lambda \to \infty$ in (3.6) and subtract the singularity in $(1-\alpha)$. That this subtraction is independent of a, in general, is the content of the theorem of t'Hooft and Veltman (1972). In both cases the logarithmic infrared singularity of $I_\alpha(a)$, as a function of a, for $\alpha = 1$, is preserved. That is

$$I_1(a) \sim \ln a.$$

In both cases the subtraction is independent of a (see the discussion in § 4).

There are two ways of carrying out renormalization. The first is the one which fixes the values of some vertex functions at some given momentum, the second is minimal renormalization to which we return in § 4 below.

The determination of the vertex functions, which have to be treated by renormalization, is technically the same as in a discussion of ultraviolet divergences. We have to determine which vertex functions have primitive 'logarithmic' integrands. From the discussion in § 3.3, above, we know the primitive difference between the number of powers in the numerator and denominator at $\alpha = 6$, for a function with E external legs, is

$$\delta = 6-2E.$$

For $E = 2$ and 3 this difference is positive. But, for a dimensionally regularized theory with zero mass, $E = 2$ has logarithmic integrands as well. Thus we have to renormalize $\Gamma^{(2)}$ and $\Gamma^{(3)}$. In order to calculate the exponent ν, vertex functions with the composite operator ϕ^2 have also to be studied. From these, only $\Gamma^{(2,1)}$—corresponding to $\langle\phi\phi\phi^2\rangle$—has primitive logarithmic integrands.

The central theorem on renormalization, which we adopt, states that it is sufficient to subtract the poles of the functions with primitive logarithmic integrals, in order to make all vertex functions finite.

4. The renormalization procedure

The theory defined in § 2 with $g_{40} = 0$, is considered at zero mass, i.e. at its critical point. The reason for considering a zero mass theory was discussed in § 1. The temperature dependence is incorporated by expanding every correlation function in a power series in

$$\Delta m_0^2 = m_0^2 = m_{0c}^2 \tag{4.1}$$

where m_{0c}^2 is the critical temperature relative to the mean field transition temperature for $g_{30} = 0$. This procedure—an expansion in a 'soft operator'—has been introduced by Weinberg (1973) and was discussed in the context of phase transitions by several authors (see references 36, 37 and 38 in Brézin *et al* 1974). It simplifies the computations considerably.

The renormalization is carried out following t'Hooft (1973). Since the mass is zero and since at zero mass *the mass subtraction terms vanish* when dimensionally regularized, we need to introduce three dimensionless functions $u_0(u, \epsilon)$, $Z_\phi(u, \epsilon)$,

$Z_{\phi^2}(u, \epsilon)$ where

$$g_{30} = \mu^{\epsilon/2} u_0 \equiv g_0. \tag{4.2}$$

μ is an arbitrary momentum scale, and

$$\epsilon = 6 - d. \tag{4.3}$$

The functions Z_ϕ, Z_{ϕ^2} and u_0 are determined by requiring that

$$Z_\phi \Gamma_B^{(2)}(p, -p; g_0) = \Gamma^{(2)}(p, -p; u, \epsilon, \mu) \tag{4.4}$$

$$Z_\phi^{3/2} \Gamma_B^{(3)}(p_1, p_2, p_3; g_0) = \Gamma^{(3)}(p_1, p_2, p_3; u, \epsilon, \mu) \tag{4.5}$$

$$Z_{\phi^2} \Gamma_B^{(2,1)}(p_1, p_2; p_3; g_0) = \Gamma^{(2,1)}(p_1, p_2; p_3; u, \epsilon, \mu) \tag{4.6}$$

be finite as $\epsilon \to 0$ order by order in u. The Γ_B are dimensionally regularized. This ensures that all other vertices of many ϕ and many ϕ^2 are finite, provided that one multiplies the bare functions by $Z_\phi^{1/2}$ for every ϕ and by $Z_{\phi^2} Z_\phi^{-1}$ for every ϕ^2, and substitutes $\mu^\epsilon u_0(u, \epsilon)$ for g_{30}.

The functions $\Gamma^{(E)}$ and $\Gamma^{(E,M)}$ satisfy renormalization group equations which express the fact that Γ_B are independent of μ, when g_0 is kept fixed. The typical equation would be

$$\left(\mu \frac{\mu}{\partial \mu} + \beta(u, \epsilon) \frac{\partial}{\partial u} - \frac{1}{2} E \gamma_\phi(u, \epsilon)\right) \Gamma^{(E)} = 0. \tag{4.7}$$

where

$$\beta(u, \epsilon) = -\tfrac{1}{2} \epsilon (\partial \ln u_0 / \partial u)^{-1} \tag{4.8}$$

$$\gamma_\phi(u, \epsilon) = \beta(u, \epsilon)(\partial \ln Z_\phi / \partial u) \tag{4.9}$$

and analogously

$$\gamma_{\phi^2}(u, \epsilon) = \beta(u, \epsilon)(\partial \ln Z_{\phi^2} / \partial u). \tag{4.10}$$

The critical exponents are obtained by first finding the zeros of β, $u^*(\epsilon)$, which are the fixed points, and substituting them in γ_ϕ and γ_{ϕ^2}. Then, as explained in detail by Brézin *et al* (1974) one finds

$$\eta(\epsilon) = \gamma_\phi(u^*, \epsilon) \tag{4.11}$$

$$\nu^{-1} - 2 = \gamma_{\phi^2}(u^*, \epsilon) - \eta(\epsilon). \tag{4.12}$$

The problem then reduces to the determination of Z_ϕ, Z_{ϕ^2} and u_0. Usually, this is done by imposing normalization conditions (Brézin *et al* 1974, Di Castro 1972, Jegerlehmer and Schroer 1973), which prescribe definite values to the functions on the right-hand side of equations (4.4)–(4.6) at some arbitrarily chosen values of the external momenta. The disadvantage is that in the process of calculating the power series for the three functions one has to calculate complicated integrals which depend on the normalization point in momentum space. It turns out (t'Hooft and Veltman 1972, t'Hooft 1973) that they all cancel in the final expressions for Z_ϕ, Z_{ϕ^2} and u_0.

Furthermore, the functions β, γ_ϕ and γ_{ϕ^2}, for a given theory, do not have a unique power expansion in u and ϵ. If the renormalization is chosen wisely (t'Hooft 1973), they reach a very simple form which implies further cancellations. Both types of cancellation mentioned above provide powerful checks on the combinatorial and tensorial coefficients in the graphs at intermediate stages of the computation.

As was pointed out by t'Hooft (1973), if the theory is renormalizable then it is sufficient to demand that in the functions on the right-hand side of (4.4)–(4.6) only singular terms in ϵ be cancelled at every order in u. This suffices to determine Z_ϕ, Z_{ϕ^2} and u_0 uniquely.

The lowest order terms in $\Gamma_B^{(2)}$, $\Gamma_B^{(3)}$ and $\Gamma_B^{(2,1)}$ are not singular as $\epsilon \to 0$. Hence one writes

$$Z_\phi = \sum_{i=0}^{\infty} s_i u^i \tag{4.13}$$

$$Z_{\phi^2} = \sum_{i=0}^{\infty} r_i u^i \tag{4.14}$$

$$u_0 = u \sum_{i=0}^{\infty} t_i u^i \tag{4.15}$$

and then $s_0 = r_0 = t_0 = 1$. The one loop corrections are proportional to ϵ^{-1} multiplied by a term which is independent of the momentum, and then by a term which is finite when $\epsilon \to 0$ but proportional to $\ln k^2$. The procedure is carried out without making any explicit reference to the value of the external momenta. The first corrections to (4.13) will be proportional to ϵ^{-1}, and give no reason to suspect that they may depend on the external momenta.

But, when the new values of Z_ϕ, etc, are introduced, the ϵ^{-1} term in Z_ϕ will multiply $\ln k^2$ in the one loop correction to the bare functions, and thus a momentum dependent term, singular in ϵ is generated. t'Hooft and Veltman prove that this type of term will inevitably be cancelled by the two loop contribution, if the theory is renormalizable in the traditional way. No momentum dependence will enter Z_ϕ, Z_{ϕ^2} and u_0.

From the construction of the three renormalization functions it is clear that, apart from the lowest order term, all terms will be singular as $\epsilon \to 0$. On the other hand, since the functions β, γ_ϕ and γ_{ϕ^2} can be expressed in terms of renormalized vertices they are all finite in this limit (Callan 1970, Symanzik 1970). Inserting (4.15) in expression (4.8) for β, one concludes that the singularities in $(\partial \ln u_0/\partial u)^{-1}$ can only be simple poles, so that they can be compensated by the explicit ϵ in (4.8). Thus higher order poles in ϵ, which do appear in u_0, *have* to cancel. Moreover, with the exception of the lowest order term in u_0, there are no terms which are regular in ϵ. Thus β will have the form

$$\beta(u, \epsilon) = -\tfrac{1}{2}\epsilon u + b(u). \tag{4.16}$$

Similar arguments applied to γ_ϕ and γ_{ϕ^2} imply that these two functions have no explicit ϵ dependence at all (Gross 1975) which implies many more cancellations of the pole terms in Z_ϕ and Z_{ϕ^2}. The final ϵ dependence of the critical exponents is due to the fact that one inserts the values of u at the fixed point in the power series.

In fact the above considerations imply that one could equally well calculate β, γ_ϕ and γ_{ϕ^2} (Gross 1975) for $\epsilon = 0$ directly and then simply add the term $(-\epsilon u/2)$ to β. We prefer the longer way since it provides checks.

5. Renormalization of the Potts model

In the model described in § 2, with $g_{40} = 0$, there are n components, and so, in principle, the vertices which have to be renormalized are tensors in the component indices. However, the only tensors with two and three indices, which are invariant under the

hyper-tetrahedral symmetry, are δ_{ij} and Q_{ijk}. respectively. The two point function is proportional to δ_{ij}, and the three point function to Q_{ijk}. Also since the ϕ^2 insertion which makes $\Gamma^{(2)}$ into $\Gamma^{(2,1)}$ is really

$$\tfrac{1}{2}\boldsymbol{\phi}^2 = \tfrac{1}{2}\sum_i \phi_i^2 \tag{5.1}$$

$\Gamma^{(2,1)}$ is also proportional to δ_{ij}. Thus the functions Z_ϕ, Z_{ϕ^2} and u_0 are scalars in the component space.

In order to calculate the critical exponents to second order in ϵ we have to calculate the renormalization functions to second order in u^2, beyond the lowest order term. The coefficients s_2, r_2 and t_2 have to be calculated to $O(\epsilon^0)$; they start with $O(\epsilon^{-1})$. The coefficients s_4, r_4, t_4 have to be calculated to $O(\epsilon^{-1})$; they start at $O(\epsilon^{-2})$. The odd coefficients vanish.

If we write†

$$\Gamma_b^{(2)}(p) = p^2(1 - a_1 u_0^2 - a_2 u_0^4) \tag{5.2}$$

$$\Gamma_b^{(3)}(p) = u_0(1 + b_1 u_0^2 + b_2 u_0^4) \tag{5.3}$$

$$\Gamma_b^{(2,1)}(p) = 1 + c_1 u_0^2 + c_2 u_0^4 \tag{5.4}$$

where the tensors have been removed, then by the procedure of minimal renormalization, as described in the previous section, we find:

$$s_2 = [a_1]_s \tag{5.5}$$

$$r_2 = [-c_1]_s \tag{5.6}$$

$$t_2 = [-(\tfrac{3}{2}s_2 + b_1)]_s \tag{5.7}$$

$$s_4 = [s_2 a_1 + a_2 + 2a_1 t_2]_s \tag{5.8}$$

$$r_4 = [-(r_2 c_1 + c_2 + 2c_1 t_2)]_s \tag{5.9}$$

$$t_4 = [-\tfrac{3}{2}(s_4 + \tfrac{1}{4}s_2^2) - \tfrac{3}{2}s_2(t_2 + b_1) - 3b_1 t_2 - b_2]_s. \tag{5.10}$$

The notation $[A]_s$ means 'the singular part of A, as a function of ϵ'. These equations follow simply from the requirement that in equations (4.4)–(4.6) the singularities in ϵ cancel, order by order in u.

In terms of the s_i, r_i and t_i one can write β, γ_ϕ and γ_{ϕ^2} using (4.8)–(4.10). The result is:

$$\beta(u) = -\tfrac{1}{2}\epsilon u[1 - 2t_2 u^2 - (4t_4 - 6t_2^2)u^4] \tag{5.11}$$

$$\gamma_\phi(u) = -\epsilon u^2[s_2 + (2s_4 - s_2^2 - 2s_2 t_2)u^2] \tag{5.12}$$

$$\gamma_{\phi^2}(u) = -\epsilon u^2[r_2 + (2r_4 - r_2^2 - 2r_2 t_2)u^2]. \tag{5.13}$$

The cancellations we have discussed in the previous section imply that momentum dependent terms have to cancel identically, in s_4, r_4 and t_4 independently. Then, the terms proportional to ϵ^{-2} in the internal parentheses in β, γ_ϕ and γ_{ϕ^2} have to cancel as well.

† The coupling constants u_0 and u and $\Gamma^{(3)}$ have been redefined to include the angular factors: $u^2[2^{d-1}\pi^d\Gamma(\frac{1}{2}d)] \to u^2$.

Given (5.11), (5.12) and (5.13) we can calculate u, η, and ν^{-1} in terms of the s, r and t. The result is:

$$u^{*2} = \frac{1}{2t_2} - \frac{(4t_4 - 6t_2^2)}{(2t_2)^3} \tag{5.14}$$

$$\eta = -\frac{s_2}{2t_2}\epsilon - \frac{2t_2(2s_4 - s_2^2 - 2s_2t_2) - s_2(4t_4 - 6t_2^2)}{(2t_2)^3}\epsilon \tag{5.15}$$

$$\nu^{-1} - 2 + \eta = -\frac{r_2}{2t_2}\epsilon - \frac{2t_2(2r_4 - r_2^2 - 2r_2t_2) - r_2(4t_4 - 6t_2^2)}{(2t_2)^3}\epsilon. \tag{5.16}$$

Recall that each of the factors appearing in the above expression is proportional to ϵ^{-1}.

6. The critical exponents for general n

Despite the fact that most of the detailed calculations are relegated to the appendix, we discuss the coefficients of the graphs which follow from the tensorial structure of the interaction here, since these coefficients serve in the definition of the results.

There are three functions of n, the number of components, from which all the coefficients can be constructed. They are related to the graphs shown in figure 1.

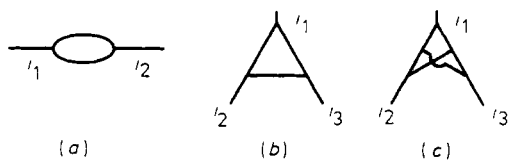

Figure 1. Graphs defining the three independent tensorial coefficients.

The contributions of the graphs are:

$$(a) \qquad \tfrac{1}{2}A_1 \sum_{ijk} Q_{i_1jk}Q_{jki_2} = \tfrac{1}{2}A_1(n+1)^2(n-1)\delta_{i_1i_2} \equiv \tfrac{1}{2}A_1\alpha_1\delta_{i_1i_2} \tag{6.1}$$

$$(b) \qquad b_1 \sum_{jkl} Q_{i_1jk}Q_{i_2ji}Q_{i_3ik} = B_1(n+1)^2(n-2)Q_{i_1i_2i_3} \equiv B_1\beta_1 Q_{i_1i_2i_3} \tag{6.2}$$

$$(c) \qquad B_4 \sum_{j_1 \ldots j_6} Q_{i_1j_1j_2}Q_{j_1j_3j_5}Q_{i_2j_3j_4}Q_{j_4j_5j_6}Q_{i_3j_5j_6}$$

$$= B_4(n+1)^6\left(1 - \frac{6}{n+1} + \frac{10}{(n+1)^2}\right)Q_{i_1i_2i_3} \equiv B_4\beta_4 Q_{i_1i_2i_3} \tag{6.3}$$

where A_1, B_1 and B_4 are the integrals corresponding to the graphs. They are discussed in the appendix. Equations (6.1)–(6.3) define α_1, β_1 and β_4.

Figure 1(a) represents a_1 in equation (5.2). There are two terms in a_2 given by figures 2(b) and 2(c) respectively.

$$a_{21} = \tfrac{1}{2}\alpha_1^2 A_2 \tag{6.4}$$

$$a_{22} = \tfrac{1}{2}\alpha_1\beta_1 A_3. \tag{6.5}$$

$$a_2 = a_{21} + a_{22}$$

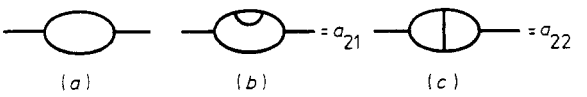

Figure 2. Graphs of $\Gamma^{(2)}$.

Figure 1(b) gives b_1 in equation (5.3). There are three terms in b_2, represented by figures 3(b), 3(c) and 3(d). The corresponding expressions are:

$$b_{21} = 3\beta_1^2 B_2 \tag{6.6}$$

$$n_{22} = \tfrac{3}{2}\alpha_1\beta_1 B_3 \tag{6.7}$$

$$b_{23} = \tfrac{1}{2}\beta_4 B_4. \tag{6.8}$$

$$b_2 = b_{21} + b_{22} + b_{23}$$

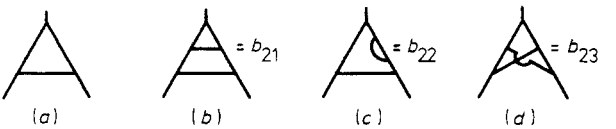

Figure 3. Graphs of $\Gamma^{(3)}$.

The terms of $\Gamma^{(2,1)}$ are given in terms of tensorial factors of $\Gamma^{(2)}$ and integrals of $\Gamma^{(3)}$. Thus, if the ϕ^2 insertion is denoted by $\odot$, then the graphs entering $\Gamma^{(2,1)}$ can be drawn as shown in Figure 4.

$$c_1 = \alpha_1 B_1 \tag{6.9}$$

$$c_{21} = \alpha_1^2 B_2 \tag{6.10}$$

$$c_{22} = 2a_1\beta_1 B_2 \tag{6.11}$$

$$c_{23} = \alpha_1^2 B_3 \tag{6.12}$$

$$c_{24} = \tfrac{1}{2}\alpha_1^2 B_3 \tag{6.13}$$

$$c_{25} = \tfrac{1}{2}\alpha_1\beta_1 B_4. \tag{6.14}$$

$$c_2 = \sum_1^5 c_{2i}$$

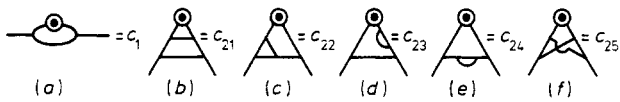

Figure 4. Graphs of $\Gamma^{(2,1)}$.

In terms of the α and β the results for the expansion coefficients of the renormalization constants can be written as

$$s_2 = -\alpha_1/6\epsilon \tag{6.15}$$

$$r_2 = -\alpha_1/\epsilon \tag{6.16}$$

$$t_2 = \frac{1}{\epsilon}\left(\frac{\alpha_1}{4} - \beta_1\right) \tag{6.17}$$

$$s_4 = \frac{1}{\epsilon^2}\left(-\frac{\alpha_1^2}{36} + \frac{\alpha_1\beta_1}{6}\right) + \frac{1}{\epsilon}\left(\frac{11}{432}\alpha_1^2 - \frac{1}{18}\alpha_1\beta_1\right) \tag{6.18}$$

$$r_4 = \frac{1}{\epsilon^1}\left(\frac{\alpha_1^2}{4} + \alpha_1\beta_1\right) + \frac{1}{\epsilon}\left(\frac{1}{48}\alpha_1^2 - \frac{1}{2}\alpha_1\beta_1\right) \tag{6.19}$$

$$t_4 = \frac{1}{\epsilon^2}\frac{3}{2}\left(\frac{\alpha_1}{4} - \beta_1\right)^2 - \frac{1}{\epsilon}\left(\frac{11}{288}\alpha_1^2 - \frac{11}{48}\alpha_1\beta_1 + \frac{3}{8}\beta_1^2 + \frac{1}{4}\beta_4\right). \tag{6.20}$$

Inserting (6.15) through (6.20) in (5.14) through (5.16) leads to:

$$u^{*2} = \frac{\epsilon}{(\frac{1}{2}\alpha_1 - 2\beta_1)}\left(1 + \frac{\frac{11}{72}\alpha_1^2 - \frac{11}{12}\alpha_1\beta_1 + \frac{3}{2}\beta_1^2 + \beta_4}{(\frac{1}{2}\alpha_1 - 2\beta_1)^2}\epsilon\right) \tag{6.21}$$

$$\eta = \frac{\alpha_1\epsilon}{6(\frac{1}{2}\alpha_1 - 2\beta_1)}\left(1 + \frac{\frac{1}{36}\alpha_1\beta_1 + \frac{1}{6}\beta_1^2 + \beta_4}{(\frac{1}{2}\alpha_1 - 2\beta_1)^2}\epsilon\right) \tag{6.22}$$

$$\nu^{-1} - 2 + \eta = \frac{\alpha_1\epsilon}{(\frac{1}{2}\alpha_1 - 2\beta_1)}\left(1 + \frac{\frac{19}{144}\alpha_1^2 + \frac{1}{3}\alpha_1\beta_1 - \frac{1}{2}\beta_1^2 + \beta_4}{(\frac{1}{2}\alpha_1 - 2\beta_1)^2}\epsilon\right). \tag{6.23}$$

7. Discussion of the results for general n

All three quantities in equations (6.12)–(6.23) have in their denominators the combination

$$w = \tfrac{1}{2}\alpha_1 - 2\beta_1.$$

Using (6.1) and (6.2) we find that

$$w = \tfrac{1}{2}(n+1)^2(7-3n). \tag{7.1}$$

This indicates that as $n \to 7/3$ (number of states $\to 10/3$) from below the fixed point disappears since $(u^*)^2$ cannot be negative. There is no fixed point for $n > 7/3$.

But as n approaches $7/3$, $(u^*)^2$ increases, and therefore the calculation becomes meaningless well before it reaches this value. We must have

$$7 - 3n > O(\epsilon^{\sigma}),$$

with $\sigma < 1$, for the results to make sense.

To determine how close n can approach its critical value we return to equation (5.11), which has the form:

$$\beta(u) = -\tfrac{1}{2}\epsilon u[1 - \tfrac{1}{2}(n+1)^2(7-3n)u^2 + Au^4] \tag{7.2}$$

with $A(n) > 0$ when $n = 7/3$. In order for equation (7.2) to have a solution, $(u)^2 > 0$, we must have:

$$\tfrac{1}{2}(n+1)^2(7-3n) > Au^2. \tag{7.3}$$

Substituting u^2 in first order in ϵ gives the condition:

$$7 - 3n > A'\epsilon^{1/2}. \tag{7.4}$$

This is a rather troubling feature. For any physical system $\epsilon \geqslant 3$. Hence, if the fixed point is to be interpreted as a continuous transition, one is restricted to very small values of n. In fact, below $n = 2$ the only case of interest is $n = 0$, the percolation problem to which we now turn.

8. The percolation problem

If we assume that $n = 0$ is beyond the difficulties caused by the appearance of a critical number of components, which depends on the number of space dimensions, then we can calculate the exponents for the percolation problem.

To second order in ϵ, the exponents which are calculated directly are:

$$\eta = -\frac{1}{21}\epsilon - \frac{206}{3^3.7^3}\epsilon^2, \tag{8.1}$$

$$\nu^{-1} = 2 - \frac{5}{21}\epsilon - \frac{653}{2.3^3.7^3}\epsilon^2. \tag{8.2}$$

The rest are obtained by using scaling relations. We find

$$\gamma = (2-\eta)\nu = 1 + \frac{1}{7}\epsilon + \frac{565}{2^2.3^2.7^3}\epsilon^2 \tag{8.3}$$

$$\beta = \frac{1}{2}(d\nu - \gamma) = 1 - \frac{1}{7}\epsilon - \frac{61}{2^2.3^2.7^3}\epsilon^2. \tag{8.4}$$

These results agree to first order with Priest and Lubensky (1975). The second order terms are somewhat different. Consequently our γ, calculated to order ϵ^2, agrees with the numerical values of Kirkpatrick, to within the errors reported there, at four and five dimensions. The second order terms in β are very small and the results are very unsatisfactory. Our results together with those of Kirkpatrick (1975) are shown in figure 5.

We conjecture that the contrast between the agreement of γ and β with the numerical results, originates from the fundamental structure of the model. It is possible that even if a first order transition has removed the foundation from the zero-mass computation, the relation between γ, η and ν in the metastable symmetric state will hold. On the other hand, in such a situation the relation between these exponents and the one describing the order parameter β, is doubly suspect.

9. Comparison with Priest and Lubensky

The representation chosen by Priest and Lubensky is somewhat different from ours. A translation can be made by replacing our interaction u^2 by t^2:

$$u^2 = 288\left(\frac{1}{n+1}\right)^3 t^2 \equiv Rt^2. \tag{9.1}$$

Then one has to replace our α_1 by a_1 using

$$\alpha_1 = 2R^{-1}a_1. \tag{9.2}$$

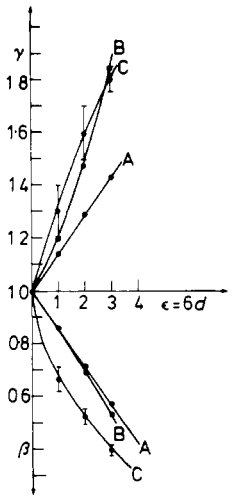

Figure 5. γ and β against $\epsilon = 6 - d$, for the $n = 0$ case. Curve A, linear in ϵ; curve B, order ϵ^2; curve C, Monte Carlo.

The coefficient β_1 has to be replaced by a_2 using:

$$\beta_1 = R^{-1} a_2 \tag{9.3}$$

and finally, β_4 is to be replaced by a_3:

$$\beta_4 = R^{-2} a_3 \tag{9.4}$$

where a_1, a_2 and a_3 is the notation employed by Priest and Lubensky.

Using (9.2)–(9.4) we can rewrite the expressions for η and $\nu^{-1} - 2 + \eta$ in terms of the a_i. We find:

$$\eta = -\frac{1}{3}\left(\frac{a_1}{2a_2 - a_1}\right)\epsilon - \frac{a_1}{54}\frac{3a_2^2 + a_1 a_2 + 36a_3}{(2a_2 - a_1)^3}\epsilon^2 \tag{9.5}$$

$$\nu^{-1} - 2 + \eta = -\frac{2a_1}{2a_2 - a_1}\epsilon + \frac{a_1}{(2a_2 - a_1)^3}\left(-\frac{19}{18}a_1^2 - a_2^2 + \frac{4}{3}a_1 a_2 - 4a_3\right)\epsilon^2. \tag{9.6}$$

10. Epilogue

The differences between our original calculation and that of Priest and Lubensky were discussed at length with Dr Lubensky. The initial result of the discussion was that I had to concede that the graphs (c) in figure 1 and (d) in figure 4, had a symmetry factor of $\frac{1}{2}$, rather than 1. This was a rather painful realization, since a superficial look at these two graphs reveals no symmetry, and then the combinations can be filled in to justify the original assertion. However, once the graph is viewed as a three dimensional figure, the plane of the three external legs appears as a plane of symmetry of reflection.

When recalculated with the corrected weights, as they appear in equations (6.8) and (6.13), our numerical values changed very little but serious differences persisted, such as the factors of 2 in front of a_3 in both η and $\nu^{-1} - 2 + \eta$. Compare equations (9.5) and

(9.6) with Priest and Lubensky's equations (5.15) and (5.16). Each of us has redone the calculation using the other's method, and apart from trivial misprints, the respective results were reproduced.

Further checks, correspondence and discussions have revealed another error in our calculation, again in the one term which is not checked intrinsically at this order. This is the $\frac{1}{2}$ in equation (A.14). An error was also found in equation (5.5) of Priest and Lubensky. When all these errors are corrected, the results of our respective calculations agree. Still γ fits numerical simulation results much better than β.

Acknowledgments

Without many discussions with Drs E Brézin and J Zinn-Justin, this work would probably have never been finished. But I would also like to mention my gratitude to Dr L Peliti for many helpful discussions. Finally, I am deeply indebted to Dr T C Lubensky for many discussions, a long correspondence and a patient comparison of very different methods.

Appendix 1. Example of the dimensional regularization in problems of critical phenomena

To illustrate the argument of § 3.1 we consider the graph in figure 6.

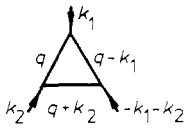

Figure 6. The one-loop graph in $\Gamma^{(3)}$.

Denoting the graph by $I(k_1, k_2)$, we have

$$I(k_1, k_2) = \int \mathrm{d}q \frac{1}{q^2(q-k_1)^2(q+k_2)^2}. \tag{A.1}$$

Introducing Feynman parameters x and y we have

$$I = \Gamma^{(3)} \int \mathrm{d}q \int \mathrm{d}x\, \mathrm{d}y \frac{1}{[q^2 - 2(xk_1 - yk_2)q + xk_1^2 + yk_2^2]^3}. \tag{A.2}$$

Remembering that a factor

$$S_d = 2^{d-1}\pi^d \Gamma(d/2)$$

was absorbed in u^2, the dimensionally regularized result for (A.2) is obtained by using equation (A.5) of t'Hooft and Veltman (1972). Namely, in our notation

$$\int \mathrm{d}q \frac{1}{[q^2 + 2kq + M^2]^\alpha} = \frac{1}{2} \frac{\Gamma(\frac{1}{2}d)\Gamma(\alpha - \frac{1}{2}d)}{\Gamma(\alpha)} (M^2 - k^2)^{\frac{1}{2}d - \alpha} \tag{A.3}$$

and

$$I(k_1, k_2) = \tfrac{1}{2}\Gamma(\tfrac{1}{2}d)\Gamma(3-\tfrac{1}{2}d) \int \mathrm{d}x\, \mathrm{d}y[x(1-x)k_1^2 + y(1-y)k_2^2 + 2xyk_1k_2]^{\frac{1}{2}d-3}. \tag{A.4}$$

On the other hand, if we want an integral which is ultraviolet convergent beyond six dimensions, so that the cut-off dependence can be disregarded, we should perform an integration by parts. This we do using t'Hooft and Veltman's 'partial p'. Namely.

$$\frac{1}{d}\sum_i \frac{\partial q_i}{\partial q_i} = 1 \tag{A.5}$$

is inserted in (A.1). The result is

$$I(k_1, k_2) = -\frac{2}{6-d}\left(\int^{\Lambda} \frac{k_1(q-k_1)}{q^2[(q-k_1)^2]^2(q+k_2)^2} - \int^{\Lambda} \frac{k_2(q+k_2)}{q^2(q-k_1)^2[(q+k_2)^2]^2}\right) + \frac{1}{d-6}S_\Lambda, \tag{A.6}$$

where $S_\Lambda(k_1, k_2)$ is the integral of $q_i/[q^2(q-k_1)^2(q+k_2)^2]$ over the surface on which $|q| = \Lambda$.

The last term in (A.6), being calculated on the surface, is a harmless function of k_1 and k_2, except for $d = 6$, But this singularity is dealt with by the renormalization, as is discussed in the text.

The two integrals in (A.6) are ultraviolet convergent, and we can let $\Lambda \to \infty$. The difference is once again harmless except for the pole at $d = 6$. These two integrals can be computed legitimately, since they are convergent, using (A.3) and

$$\int \frac{\boldsymbol{q}}{[q^2+2kq+M^2]^\alpha}\mathrm{d}\boldsymbol{q} = \frac{1}{2}\frac{\Gamma(\frac{1}{2}d)\Gamma(\alpha-\frac{1}{2}d)}{\Gamma(\alpha)}[M^2-k^2]^{\frac{1}{2}d-\alpha}(-\boldsymbol{k}) \tag{A.7}$$

(t'Hooft and Veltman, equation (A.6)). The result is (A.4).

Appendix 2. Cancellation of momentum dependent parts in minimal renormalization

In order to indicate how cancellations occur in the programme of minimal renormalization, we will use a short cut, which saves a certain amount of writing, in a calculation carried to the necessary order. The technique used by t'Hooft and Veltman (1972) is much more systematic, and makes the generalization to any number of loops automatic.

We need three terms for the calculation of $\Gamma^{(2)}(p)$. They are

$$A_1 = -\frac{p^2}{3\epsilon}\left(1 + \frac{7}{12}\epsilon - \frac{1}{2}\epsilon \ln p^2\right) \tag{A.8}$$

$$A_2 = \frac{p^2}{18\epsilon^2}\left(1 + \frac{25}{12}\epsilon - \epsilon \ln p^2\right) \tag{A.9}$$

$$A_3 = -\frac{p^2}{3\epsilon^2}\left(1 + \frac{3\epsilon}{2} - \epsilon \ln p^2\right). \tag{A.10}$$

As was discussed in the text, A_1 is calculated to O(1), while A_2 and A_3 to $O(\epsilon^{-1})$.

For $\Gamma^{(3)}(p_1, p_2, p_3)$ we need

$$B_1 = \frac{1}{\epsilon}\left(1 - \frac{3}{4}\epsilon - \epsilon L\right) \tag{A.11}$$

$$B_2 = \frac{1}{2\epsilon^2}\left(1 - \frac{5}{4}\epsilon - 2\epsilon L\right) \tag{A.12}$$

$$B_3 = -\frac{1}{6\epsilon^2}\left(1 - \frac{11}{12}\epsilon - 2\epsilon L\right) \tag{A.13}$$

$$B_4 = \frac{1}{2\epsilon}, \tag{A.14}$$

where

$$L = \int_0^1 dx \int_0^1 dy\, \theta(1-x-y) \ln\left[x(1-x)p_1^2 + y(1-y)p_2^2 + 2xyp_1p_2\right]. \tag{A.15}$$

Using (6.1) and (6.2), in conjunction with equations (A.8) and (A.11) in (5.5) and (5.7), one immediately obtains (6.15) and (6.17). No cancellations take place at this level, since there are no momentum dependent terms which are singular as $\epsilon \to 0$.

If we proceed to calculate s_4, using (5.8), we find that all the terms will contain $\epsilon^{-1} \ln p^2$. These singular momentum dependent contributions enter proportional sometimes to α_1^2, sometimes to $\alpha_1\beta_1$. Using (6.4), (6.5), (6.15) and (6.17) together with equations (A.9) and (A.10), one finds that these two contributions vanish independently, and the result for s_4 is given by (6.18).

In calculating t_4 and r_4, one comes across terms of the form $\epsilon^{-1}L$, which are singular momentum dependent terms. These again cancel identically. Terms proportional to α_1^2, $\alpha_1\beta_1$ and to β_1^2 cancel independently. It is then a matter of straightforward arithmetic to arrive at (6.20). Similarly for r_4, equation (6.19).

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